On the Existence of the Sequent-style Proof Systems

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A Negative Line of Research

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- Proposing a convincing formalization of what we mean by natural and nice proof systems,
- Finding an invariant, i.e., a property that the logic of a nice proof system enjoys,
- And finally, proving that almost all logics in a certain given category do not enjoy that property.

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- Focused axioms are just a modest generalization of the axioms of **LJ**.
- A focused rule is a rule with one main formula in its consequence such that the rule respects both the side of this main formula and the occurrence of atoms in it, i.e. if the main formula occurred in the left-side (right-side) of the consequence, all non-contextual formulas in the premises should also occur in the left-side (right-side) and if an atom occurs in these formulas, it should also occur in the main formula.

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The conjunction and disjunction rules in the intuitionistic calculus **LJ** can be considered as examples of focused rules. But implication rules are not focused.

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- The invariant is uniform interpolation,
- Only seven of the super-intuitionistic logics have uniform interpolation.

Our Contribution

In this talk we will present a second approximation for *nice proof systems*. First we define the semi-analytic rules as our candidate for the natural well-behaved sequent-style rules. These rules can be defined roughly as the focused rules relaxing the side preserving condition. Therefore, they cover a vast variety of rules including focused rules, implication rules, non-context sharing rules in substructural logics and so many others. We also consider the usual modal rules of K, D, $K4$, $K4D$ and $S4$. Then we show:

Main Result (informal.)

Theorem (A., Jalali)

(*i*) If a *sufficiently strong sub-structural logic* has a sequent-style proof system only consisting of *semi-analytic* rules and focused axioms, it has the Craig interpolation property. As a result, many substructural logics and all super-intuitionistic logics, except seven of them, do not have a sequent calculus of the mentioned form.

(*ii*) If a *sufficiently strong sub-structural logic* has a terminating sequent-style proof system only consisting of semi-analytic rules and focused axioms, it has the uniform interpolation property. Consequently, **K4** and **S4** do not have a terminating sequent calculus of the mentioned form.

Main Result (informal.)

Note that the three steps in our result are:

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- Our nice proof systems are semi-analytic proof systems i.e., the systems consisting of the suitable variants of semi-analytic rules and focused axioms,
- The invariants are both Craig and uniform interpolation,
- The class of logics is the class of all extensions of a very basic substructural logic. Therefore, besides the known results on the lack of interpolation in super-intuitionistic and modal logics we can also use some negative results for some sub-structural logics [4].

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- The invariants are both Craig and uniform interpolation,
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Hence, we have to first explain what we mean by a semi-analytic rule and then we have to introduce the logics for which the Craig interpolation fails to hold.

Preliminaries: Basic Sub-structural Logics

$$\frac{}{\phi \Rightarrow \phi} \qquad \frac{}{\Rightarrow 1} \qquad \frac{}{0 \Rightarrow}$$

$$\frac{\Gamma \Rightarrow \Delta}{\Gamma, 1 \Rightarrow \Delta} L1 \qquad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow 0, \Delta} R0$$

$$\frac{\Gamma, \phi \Rightarrow \Delta}{\Gamma, \phi \wedge \psi \Rightarrow \Delta} L\wedge \qquad \frac{\Gamma \Rightarrow \phi, \Delta \qquad \Gamma \Rightarrow \psi, \Delta}{\Gamma \Rightarrow \phi \wedge \psi, \Delta} R\wedge$$

$$\frac{\Gamma, \phi \Rightarrow \Delta \qquad \Gamma, \psi \Rightarrow \Delta}{\Gamma, \phi \vee \psi \Rightarrow \Delta} L\vee \qquad \frac{\Gamma \Rightarrow \phi, \Delta}{\Gamma \Rightarrow \phi \vee \psi, \Delta} R\vee \qquad \frac{\Gamma \Rightarrow \psi, \Delta}{\Gamma \Rightarrow \phi \vee \psi, \Delta} R\vee$$

$$\frac{\Gamma, \phi, \psi \Rightarrow \Delta}{\Gamma, \phi * \psi \Rightarrow \Delta} L* \qquad \frac{\Gamma \Rightarrow \phi, \Delta \qquad \Sigma \Rightarrow \psi, \Lambda}{\Gamma, \Sigma \Rightarrow \phi * \psi, \Delta, \Lambda} R*$$

$$\frac{\Gamma \Rightarrow \phi, \Delta \qquad \Sigma, \psi \Rightarrow \Lambda}{\Gamma, \Sigma, \phi \to \psi \Rightarrow \Delta, \Lambda} L\to \qquad \frac{\Gamma, \phi \Rightarrow \psi, \Delta}{\Gamma \Rightarrow \phi \to \psi, \Delta} R\to$$

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- If we also add the single-conclusion version of the following axioms, we will have the system $\mathbf{FL_e}$.

$$\frac{}{\Gamma \Rightarrow \top, \Delta} \qquad \frac{}{\Gamma, \bot \Rightarrow \Delta}$$

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$$\frac{}{\Gamma \Rightarrow \top, \Delta} \qquad \frac{}{\Gamma, \bot \Rightarrow \Delta}$$

- In the multi-conclusion case define $\mathbf{CFL_e}^-$ and $\mathbf{CFL_e}$ with the same rules as $\mathbf{FL_e}^-$ and $\mathbf{FL_e}$, this time in their full multi-conclusion version and add $+$ to the language and the following rules to the systems:

$$\frac{\Gamma, \phi \Rightarrow \Delta \qquad \Sigma, \psi \Rightarrow \Lambda}{\Gamma, \Sigma, \phi + \psi \Rightarrow \Delta, \Lambda} \; L+ \qquad \frac{\Gamma \Rightarrow \phi, \psi, \Delta}{\Gamma \Rightarrow \phi + \psi, \Delta} \; R+$$

Preliminaries: Structural Rules

Weakening rules:

$$\frac{\Gamma \Rightarrow \Delta}{\Gamma, \phi \Rightarrow \Delta}\ Lw \qquad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \phi, \Delta}\ Rw$$

Contraction rules:

$$\frac{\Gamma, \phi, \phi \Rightarrow \Delta}{\Gamma, \phi \Rightarrow \Delta}\ Lc \qquad \frac{\Gamma \Rightarrow \phi, \phi, \Delta}{\Gamma \Rightarrow \phi, \Delta}\ Rc$$

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Semi-analytic rules: Single-conclusion

- *Left semi-analytic rule*:

$$\frac{\langle\langle \Pi_j, \bar{\psi}_{js} \Rightarrow \bar{\theta}_{js} \rangle_s \rangle_j \qquad \langle\langle \Gamma_i, \bar{\phi}_{ir} \Rightarrow \Delta_i \rangle_r \rangle_i}{\Pi_1, \cdots, \Pi_m, \Gamma_1, \cdots, \Gamma_n, \phi \Rightarrow \Delta_1, \cdots, \Delta_n}$$

where Π_j, Γ_i and Δ_i's are meta-multiset variables and

$$\bigcup_{i,r} V(\bar{\phi}_{ir}) \cup \bigcup_{j,s} V(\bar{\psi}_{js}) \cup \bigcup_{j,s} V(\bar{\theta}_{js}) \subseteq V(\phi)$$

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- *Right semi-analytic rule*:

$$\frac{\langle\langle \Gamma_i, \bar{\phi}_{ir} \Rightarrow \bar{\psi}_{ir} \rangle_r \rangle_i}{\Gamma_1, \cdots, \Gamma_n \Rightarrow \phi}$$

  where Γ_i's are meta-multiset variables and

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Semi-analytic rules: Context-sharing

Context-sharing semi-analytic:

$$\frac{\langle\langle \Gamma_i, \bar{\psi}_{is} \Rightarrow \bar{\theta}_{is} \rangle_s \rangle_i \qquad \langle\langle \Gamma_i, \bar{\phi}_{ir} \Rightarrow \Delta_i \rangle_r \rangle_i}{\Gamma_1, \cdots, \Gamma_n, \phi \Rightarrow \Delta_1, \cdots, \Delta_n}$$

where Γ_i and Δ_i's are meta-multiset variables and

$$\bigcup_{i,r} V(\bar{\phi}_{ir}) \cup \bigcup_{i,s} V(\bar{\psi}_{is}) \cup \bigcup_{i,s} V(\bar{\theta}_{is}) \subseteq V(\phi)$$

We will call the conditions for the variables in all the semi-analytic rules, the *occurrence preserving* conditions.
Note that in the left rule, for each i we have $|\Delta_i| \leqslant 1$, and since the size of the succedent of the conclusion of the rule must be at most 1, it means that at most one of Δ_i's can be non-empty.

- *Left multi-conclusion semi-analytic rule*:

$$\frac{\langle\langle \Gamma_i, \bar{\phi}_{ir} \Rightarrow \bar{\psi}_{ir}, \Delta_i \rangle_r \rangle_i}{\Gamma_1, \cdots, \Gamma_n, \phi \Rightarrow \Delta_1, \cdots, \Delta_n}$$

  with the same occurrence preserving condition as above and the same condition that all Γ_i's and Δ_i's are meta-multiset variables.

Semi-analytic rules: Multi-conclusion

- *Left multi-conclusion semi-analytic rule*:

$$\frac{\langle\langle \Gamma_i, \bar{\phi}_{ir} \Rightarrow \bar{\psi}_{ir}, \Delta_i \rangle_r \rangle_i}{\Gamma_1, \cdots, \Gamma_n, \phi \Rightarrow \Delta_1, \cdots, \Delta_n}$$

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- *Right multi-conclusion semi-analytic rule*:

$$\frac{\langle\langle \Gamma_i, \bar{\phi}_{ir} \Rightarrow \bar{\psi}_{ir}, \Delta_i \rangle_r \rangle_i}{\Gamma_1, \cdots, \Gamma_n \Rightarrow \phi, \Delta_1, \cdots, \Delta_n}$$

  again with the similar occurrence preserving condition and the same condition that all Γ_i's and Δ_i's are meta-multiset variables.

Semi-analytic rules

By a *modal rule*, we mean one of the following usual modal rules:

$$\frac{\Gamma \Rightarrow \phi}{\Box\Gamma \Rightarrow \Box\phi}\, K \quad \frac{\Gamma \Rightarrow}{\Box\Gamma \Rightarrow}\, D \quad \frac{\Box\Gamma \Rightarrow \phi}{\Box\Gamma \Rightarrow \Box\phi}\, RS4 \quad \frac{\Gamma, \phi \Rightarrow \Delta}{\Gamma, \Box\phi \Rightarrow \Delta}\, LS4$$

with the conditions that Γ and Δ are meta-multiset variables, ϕ is a meta-formula variable, whenever the rule (D) is present, the rule (K) must be present, and similarly whenever the rule $(RS4)$ is present in a system, the rule $(LS4)$ must be present, as well.

Generic Examples

Example

A generic example of a left semi-analytic rule is the following:

$$\frac{\Gamma, \phi_1, \phi_2 \Rightarrow \psi \qquad \Gamma, \theta \Rightarrow \eta \qquad \Pi, \mu_1, \mu_2, \mu_3 \Rightarrow \Delta}{\Gamma, \Pi, \alpha \Rightarrow \Delta}$$

where

$$V(\phi_1, \phi_2, \psi, \theta, \eta, \mu_1, \mu_2, \mu_3) \subseteq V(\alpha)$$

and a generic example of a context-sharing left semi-analytic rule is:

$$\frac{\Gamma, \theta \Rightarrow \eta \qquad \Gamma, \mu_1, \mu_2, \mu_3 \Rightarrow \Delta}{\Gamma, \alpha \Rightarrow \Delta}$$

where

$$V(\theta, \eta, \mu_1, \mu_2, \mu_3) \subseteq V(\alpha)$$

Concrete Examples

Example

For some concrete examples, note that all the usual conjunction, disjunction and implication rules for **IPC** are semi-analytic. The same also holds for all the rules in sub-structural logic $\mathbf{FL_e}$, the weakening and the contraction rules and some of the well-known restricted versions of them including the following rules for exponentials in linear logic:

$$\frac{\Gamma, !\phi, !\phi \Rightarrow \Delta}{\Gamma, !\phi \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow \Delta}{\Gamma, !\phi \Rightarrow \Delta}$$

For a context-sharing semi-analytic rule, consider the following rule in the Dyckhoff calculus for **IPC**:

$$\frac{\Gamma, \psi \to \gamma \Rightarrow \phi \to \psi \qquad \Gamma, \gamma \Rightarrow \Delta}{\Gamma, (\phi \to \psi) \to \gamma \Rightarrow \Delta}$$

Non-examples

Example

For a concrete non-example consider the cut rule; it is not semi-analytic because it does not meet the variable occurrence condition. Moreover, the following rule in the calculus of **KC**:

$$\frac{\Gamma, \phi \Rightarrow \psi, \Delta}{\Gamma \Rightarrow \phi \to \psi, \Delta}$$

in which Δ should consist of negation formulas is not a multi-conclusion semi-analytic rule, simply because the context is not free for all possible substitutions. The rule of thumb is that any rule in which we have *side conditions on the contexts* is not semi-analytic.

Focused axioms

A sequent is called a *focused axiom* if it has the following form:

(1) Identity axiom: $(\phi \Rightarrow \phi)$

(2) Context-free right axiom: $(\Rightarrow \bar{\alpha})$

(3) Context-free left axiom: $(\bar{\beta} \Rightarrow)$

(4) Contextual left axiom: $(\Gamma, \bar{\phi} \Rightarrow \Delta)$

(5) Contextual right axiom: $(\Gamma \Rightarrow \bar{\phi}, \Delta)$

where Γ and Δ are meta-multiset variables and in (2) the variables in any pair of elements in $\bar{\alpha}$ are equal. The same condition also holds for any pair of elements in $\bar{\beta}$ in (3) or in $\bar{\phi}$ in (4) and (5). A sequent is called context-free focused axiom if it has the form (1), (2) or (3).

Focused axioms

Example

It is easy to see that the axioms given in the preliminaries are examples of focused axioms. Here are some more examples:

$$\neg 1 \Rightarrow \quad , \quad \Rightarrow \neg 0$$

$$\phi, \neg\phi \Rightarrow \quad , \quad \Rightarrow \phi, \neg\phi$$

$$\Gamma, \neg\top \Rightarrow \Delta \quad , \quad \Gamma \Rightarrow \Delta, \neg\bot$$

where the first four are context-free while the last two are contextual.

Main Result (formal.)

Theorem

(i) *If* $\mathbf{FL_e} \subseteq L$, ($\mathbf{FL_e}^- \subseteq L$) *and L has a single-conclusion sequent calculus consisting of semi-analytic rules, modal rules K, D or S4 and focused axioms (context-free focused axioms), then L has Craig interpolation.*

(ii) *If* $\mathbf{IPC} \subseteq L$ *and L has a single-conclusion sequent calculus consisting of semi-analytic rules, context-sharing semi-analytic rules, modal rules and focused axioms, then L has Craig interpolation.*

(iii) *If* $\mathbf{CFL_e} \subseteq L$, ($\mathbf{CFL_e}^- \subseteq L$) *and L has a multi-conclusion sequent calculus consisting of semi-analytic rules, modal rules K, D or S4 and focused axioms (context-free focused axioms), then L has Craig interpolation.*

Positive Application

As a positive application we have the following:

Corollary

The logics $\mathbf{FL_e}$, $\mathbf{FL_{ec}}$, $\mathbf{FL_{ew}}$, $\mathbf{CFL_e}$, $\mathbf{CFL_{ew}}$, $\mathbf{CFL_{ec}}$, **ILL**, **CLL**, **IPC**, **CPC** *and their* **K**, **KD** *and* **S4** *versions have the Craig interpolation property. The same also goes for* **K4** *and* **K4D** *extensions of* **IPC** *and* **CPC**.

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Proof.

The usual sequent calculi for these logics consist of some suitable variants of semi-analytic rules and modal rules. □

Negative Applications

Corollary

Except **IPC**, **LC**, **KC**, $\mathbf{Bd_2}$, **Sm**, **GSc** *and* **CPC**, *none of the consistent super-intuitionistic logics have a single-conclusion sequent calculus consisting only of single-conclusion semi-analytic rules, context-sharing semi-analytic rules and focused axioms.*

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Corollary

Except at most thirty seven logics, none of the extensions of **S4** *have a single-conclusion (multi-conclusion) sequent calculus consisting only of single-conclusion (multi-conclusion) semi-analytic rules, context-sharing semi-analytic rules, modal rules and focused axioms.*

Some Sub-Structural Logics

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So far, we have introduced our proposal for the nice systems and their connection to Craig interpolation. In the following we will introduce the sub-structural logics that lack the interpolation property [4].

A *pointed commutative residuated lattice* is the structure $\mathbf{A} = \langle A, \wedge, \vee, *, \rightarrow, 0, 1 \rangle$ with binary operations $\wedge, \vee, *, \rightarrow$, and constants $0, 1$ such that $\langle A, \wedge, \vee \rangle$ is a lattice with order $\leqslant$, $\langle A, *, 1 \rangle$ is a commutative monoid, and $x * y \leqslant z$ if and only if $x \leqslant y \rightarrow z$ for all $x, y, z \in A$.

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So far, we have introduced our proposal for the nice systems and their connection to Craig interpolation. In the following we will introduce the sub-structural logics that lack the interpolation property [4].

A *pointed commutative residuated lattice* is the structure $\mathbf{A} = \langle A, \wedge, \vee, *, \to, 0, 1\rangle$ with binary operations $\wedge, \vee, *, \to$, and constants $0, 1$ such that $\langle A, \wedge, \vee\rangle$ is a lattice with order $\leqslant$, $\langle A, *, 1\rangle$ is a commutative monoid, and $x * y \leqslant z$ if and only if $x \leqslant y \to z$ for all $x, y, z \in A$. Consider the following conditions for residuated lattices:

$$(prl) : 1 \leqslant (x \to y) \vee (y \to x)$$

$$(inv) : \neg\neg x = x$$

$$(bd) : 0 \leqslant x$$

$$(fp) : 0 = 1$$

$$(can) : x \to (x * y) = y$$

$$(nc) : x \wedge \neg x \leqslant 0$$

$$(dis) : x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$$

$$(int) : x \leqslant 1$$

$$(id) : x = x * x$$

$$(div) : x * (x \to y) = y * (y \to x)$$

$$(rcan) : 1 = \neg x \vee ((x \to (x * y)) \to y)$$

Some Sub-Structural Logics

Based on the conditions defined above, we can define the following logics. Note that since all of these logics have the axioms (prl) and (dis), we only mention the other axioms of the systems:

$$(UL^-)$$

$$(MTL) : (int), (bd)$$

$$(IMTL) : (int), (bd), (inv)$$

$$(G) : (int), (bd), (id)$$

$$(P) : (int), (bd), (div), (rcan)$$

$$(UML^-) : (id)$$

$$(IUML^-) : (id), (inv), (fp)$$

$$(IUL^-) : (inv)$$

$$(SMTL) : (int), (bd), (nc)$$

$$(BL) : (int), (bd), (div)$$

$$(\text{Ł}) : (int), (bd), (div), (inv)$$

$$(CHL) : (int), (fp), (div), (can)$$

$$(RM^e) : (id), (inv)$$

$$(A) : (inv), (fp), (can)$$

Some Sub-Structural Logics

Furthermore, for $n > 1$ define

$$L_n = \{0, \tfrac{1}{n-1}, \cdots, \tfrac{n-2}{n-1}, 1\} \qquad , \qquad L_\infty = [0,1]$$

and the pointed commutative residuated lattices (again for $n > 1$)

$$\mathbf{L_n} = \langle L_n, min, max, *_{Ł}, \rightarrow_{Ł}, 1, 0\rangle$$

$$\mathbf{G_n} = \langle L_n, min, max, min, \rightarrow_G, 1, 0\rangle$$

where $x *_{Ł} y = max(0, x + y - 1)$, $x \rightarrow_{Ł} y = min(1, 1 - x + y)$, and $x \rightarrow_G y$ is y if $x > y$, otherwise 1. Then, for $n > 1$, $Ł_n$ and G_n are the logics with equivalent algebraic semantics $\mathcal{V}(\mathbf{L_n})$ and $\mathcal{V}(\mathbf{G_n})$, respectively.

Some Sub-Structural Logics

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Some Sub-Structural Logics

- The logics G_∞ and $Ł_\infty$ are the Gödel logic and Łukasiewicz logic.
- R is the logic of a variety consisting of all distributive pointed commutative residuated lattices with the condition that $x * x \leqslant x$ for all x.
- Define RM^e_n as the logic of $\mathcal{V}(\mathbf{S_n})$, where:

$$\mathbf{S_{2m}} = \langle [-m, m] - \{0\}, min, max, *, \rightarrow, 1, -1 \rangle$$

$$\mathbf{S_{2m+1}} = \langle [-m, m], min, max, *, \rightarrow, 0, 0 \rangle$$

where:

$$x*y = \begin{cases} min(x, y) & \text{if } |x| = |y| \\ y & \text{if } |x| < |y| \\ x & \text{if } |y| < |x| \end{cases} \quad x \rightarrow y = \begin{cases} max(-(x), y) & \text{if } x \leqslant y \\ min(-(x), y) & \text{otherwise} \end{cases}$$

Negative Applications

As negative applications we have the following corollaries:

Corollary

None of the logics UL^-, IUL^-, MTL, SMTL, IMTL, R, BL, $Ł_\infty$, $Ł_n$ for $n \geqslant 3$, P, CHL and A have a single-conclusion sequent calculus consisting only of single-conclusion semi-analytic rules and context-free focused axioms.

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Corollary

None of the logics IUL^-, IMTL, $Ł_\infty$, $Ł_n$ for $n \geqslant 3$ and A have a single-conclusion (multi-conclusion) sequent calculus consisting only of single-conclusion (multi-conclusion) semi-analytic rules and context-free focused axioms.

Negative Applications

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The only IMTL-extension with a calculus consisting of single-conclusion (multi-conclusion) semi-analytic rules and context-free focused axioms, is **CPC**.

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Corollary

Except G, G_3 *and* **CPC**, *none of the consistent BL-extensions have a single-conclusion sequent calculus consisting only of single-conclusion semi-analytic rules and context-free focused axioms.*

Negative Application

Corollary

Except RM^e, $IUML^-$, **CPC**, RM_3^e , RM_4^e , **CPC** $\cap$ $IUML^-$, $RM_4^e \cap IUML^-$, *and* **CPC** $\cap$ RM_3^e, *none of the consistent extensions of* RM^e *have a single-conclusion (multi-conclusion) sequent calculus consisting only of single-conclusion (multi-conclusion) semi-analytic rules and context-free focused axioms. This category includes:*

(i) RM_n^e *for* $n \geqslant 5$,

(ii) $RM_{2m}^e \cap RM_{2n+1}^e$ *for* $n \geqslant m \geqslant 1$ *with* $n \geqslant 2$.,

(iii) $RM_{2m}^e \cap IUML^-$ *for* $m \geqslant 3$.

Thank You for Your Attention!

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